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Particle with spin 1 in a magnetic field on the hyperbolic plane H_2

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Abstract

There are constructed exact solutions of the quantum-mechanical equation for a spin $S = 1$ particle in 2-dimensional Riemannian space of constant negative curvature, hyperbolic plane, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels describing quantization of the motion of the vector particle in magnetic field on the 2-dimensional space H_2 has been found, nonrelativistic and relativistic equations have been solved.

1. Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3, 4]. In 1985 – 2010, a more general problem in a curved Riemannian background, hyperbolic and spherical planes, was extensively studied [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], providing us with a new system having intriguing dynamics and symmetry, both on classical and quantum levels.

Extension to 3-dimensional hyperbolic and spherical spaces was performed recently. In [25, 26, 27], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky H_3 and Riemann S_3 spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [28, 29, 30]. In the present paper, we consider a quantum-mechanical problem a particle with spin 1/2 described by the Dirac equation in 3-dimensional Lobachevsky and Riemann space models in presence of the external magnetic field.

In the present paper, we will construct exact solutions for a vector particle described by 10-dimensional Duffin–Kemmer equation in external magnetic field on the background of 2-dimensional spherical space H_2.

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10-dimensional Duffin–Kemmer equation for a vector particle in a curved space-time has the form [31]

$$\left\{ \beta^c \left[i \ (\ e^{\beta}_{(c)} \partial_\beta + \frac{1}{2} J^{ab} \gamma_{abc} \ ) \ + \ \frac{e}{\hbar c} A_{(c)} \right] - \frac{mc}{\hbar} \right\} \Psi = 0 \ , \tag{1.1}$$

where γ_{abc} stands for Ricci rotation coefficiemts, $A_a = e^{\beta}_{(a)} A_\beta$ represent tetrad components of electromagnetic 4-vector A_β; $J^{ab} = \beta^a \beta^b - \beta^b \beta^a$ are generators of 10-dimensional representation of the Lorentz group. For shortness, we use notation $e/c\hbar \Longrightarrow e, \ mc/\hbar \Longrightarrow M$.

In the space H_3 we will use the system of cylindric coordinates [32]

$$dS^2 = c^2 dt^2 - \ \cosh^2 z (dr^2 + \sinh^2 r \ d\phi^2) - dz^2 \ ,$$

$$u_1 = \cosh \, z \sinh \, r \cos \phi \ , \quad u_2 = \cosh \, z \sinh \, r \sin \phi \ ,$$

$$u_3 = \sinh \, z \ , \quad u_0 = \cosh \, z \cosh \, r \ ; \tag{1.2}$$

$$G = \{ \ r \in [0, +\infty), \ \phi \in [0, 2\pi], \ z \in (-\infty, +\infty) \ \} \ .$$

Generalized expression for electromagnetic potential for an homogeneous magnetic field in the curved model H_3 is given as follows

$$A_\phi = -2B \sinh^2 \frac{r}{2} = -B \ (\cosh r - 1) \ . \tag{1.3}$$

We will consider the above equation in presence of the field in the model H_3. Corresponding to cylindric coordinates $x^\alpha = (t, r, \phi, z)$ a tetrad can be chosen as

$$e^{\beta}_{(a)}(x) = \left| \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & \cosh^{-1} z & 0 & 0 \\ 0 & 0 & \cosh^{-1} z \ \sinh^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{1.4}$$

Taking into account relations

$$\Gamma^{r}{}_{jk} = \left| \begin{array}{ccc} 0 & 0 & \tanh \ z \\ 0 & -\sinh r \cosh r & 0 \\ \tanh \ z & 0 & 0 \end{array} \right| , \qquad \Gamma^{\phi}{}_{jk} = \left| \begin{array}{ccc} 0 & \coth \ r & 0 \\ \coth \ r & 0 & \tanh \ z \\ 0 & \tanh \ z & 0 \end{array} \right| ,$$

$$\Gamma^{z}{}_{jk} = \left| \begin{array}{ccc} -\sinh z \cosh z & 0 & 0 \\ 0 & -\sinh z \ \cosh z \ \sinh^2 r & 0 \\ 0 & 0 & 0 \end{array} \right| .$$

$$\gamma_{122} = \frac{1}{\cosh z \ \tanh r} \ , \qquad \gamma_{311} = \tanh z \ , \qquad \gamma_{322} = \tanh z \ , \tag{1.5}$$

eq. (1.1) reduces to the form

$$\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cosh z} \left(i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi - eB(\cosh r - 1) + iJ^{12} \cosh r}{\sinh r} \right) + \right.$$

$$+i\beta^3 \frac{\partial}{\partial z} - i\frac{\sinh z}{\cosh z} \left(\beta^1 J^{13} + \beta^2 J^{23}\right) - M \Bigg\} \Psi = 0 \ . \tag{1.6}$$

To separate the variables in eq. (1.5), we are to employ an explicit form of the Duffin–Kemmer matrices β^a; it will be most convenient to use so called cyclic representation [34], where the generator J^{12} is of diagonal form (we specify matrices by blocks in accordance with ($1-3-3-3$)-splitting)

$$\beta^0 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \qquad \beta^i = \left| \begin{array}{cccc} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{array} \right| , \tag{1.7}$$

where $e_i, \ e_i^t, \ \tau_i$ denote

$$e_1 = \frac{1}{\sqrt{2}}(-i, \ 0, \ i) \ , \qquad e_2 = \frac{1}{\sqrt{2}}(1, \ 0, \ 1) \ , \qquad e_3 = (0, i, 0) \ ,$$

$$\tau_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right| , \qquad \tau_2 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{array} \right| , \qquad \tau_3 = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| = s_3 \ . \tag{1.8}$$

The generator J^{12} explicitly reads

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 =$$

$$= \left| \begin{array}{cccc} (-e_1 e_2^+ + e_2 e_1^+) & 0 & 0 & 0 \\ 0 & (-\tau_1\tau_2 + \tau_2\tau_1) & 0 & 0 \\ 0 & 0 & (-e_1^+ \bullet e_2 + e_2^+ \bullet e_1) & 0 \\ 0 & 0 & 0 & (-\tau_1\tau_2 + \tau_2\tau_1) \end{array} \right| =$$

$$= -i \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{array} \right| = -iS_3. \tag{1.9}$$

2. Restriction to 2-dimensional model

Let us restrict ourselves to 2-dimensional case, spherical space H_2 (formally it is sufficient in eq. (1.5) to remove dependence on the variable z fixing its value by $z = 0$)

$$\left[i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \, \frac{i\partial_\phi - eB\,(\cosh r - 1) + iJ^{12} \cosh r}{\sinh r} - M \right] \Psi = 0 \ . \tag{2.1}$$

With the use of substitution

$$\Psi = e^{-i\epsilon t} e^{im\phi} \left| \begin{array}{c} \Phi_0(r) \\ \vec{\Phi}(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{array} \right| , \tag{2.2}$$

eq. (2.1) assumes the form (introducing notation $m + B(\cosh r - 1) = \nu(r)$)

$$\left[\epsilon \, \beta^0 + i\beta^1 \, \frac{\partial}{\partial r} - \beta^2 \, \frac{\nu(r) - \cosh r \, S_3}{\sinh r} - M \right] \left| \begin{array}{c} \Phi_0(r) \\ \vec{\Phi}(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{array} \right| = 0 \; . \tag{2.3}$$

Eq. (2.3) reads

$$\left[\epsilon \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| + i \left| \begin{array}{cccc} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{array} \right| \frac{\partial}{\partial r} - \right.$$

$$\left. - \frac{1}{\sinh r} \left| \begin{array}{cccc} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{array} \right| (\nu - \cosh r \; S_3) - M \right] \left| \begin{array}{c} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{array} \right| = 0 \; , \tag{2.4}$$

or in a block form

$$i e_1 \partial_r \vec{E} - \frac{1}{\sinh r} \; e_2 (\nu - \cosh r \, s_3) \vec{E} = M \; \Phi_0 \; ,$$

$$i\epsilon \cosh z \, \vec{E} + i\tau_1 \partial_r \vec{H} - \frac{\tau_2}{\sinh r} (\; \nu - \cosh r \, s_3 \;) \vec{H} = M \; \vec{\Phi} \; ,$$

$$-i\epsilon \cosh z \, \vec{\Phi} - i e_1^+ \partial_r \Phi_0 + \frac{\nu}{\sinh r} \, e_2^+ \Phi_0 = M \vec{E} \; ,$$

$$-i\tau_1 \partial_r \vec{\Phi} + \frac{(\nu - \cosh r \, s_3)}{\sinh r} \, \tau_2 \vec{\Phi} = M \; \vec{H} \; . \tag{2.5}$$

After separation of the variables we get

$$\gamma (\frac{\partial E_1}{\partial r} - \frac{\partial E_3}{\partial r}) - \frac{\gamma}{\sinh r} \left[(\nu - \cosh r) E_1 + (\nu + \cosh r) E_3 \right] = M \Phi_0 \; ,$$

$$+ i\epsilon \cosh z E_1 + i\gamma \frac{\partial H_2}{\partial r} + i\gamma \frac{\nu}{\sinh r} H_2 = M \Phi_1 \; ,$$

$$+ i\epsilon E_2 + i\gamma (\frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r}) - \frac{i\gamma}{\sinh r} \left[(\nu - \cosh r) H_1 - (\nu + \cosh r) H_3 \right] = M \Phi_2 \; ,$$

$$+i\epsilon E_3 + i\gamma\ \frac{\partial H_2}{\partial r} - i\gamma \frac{\nu}{\sinh r} H_2 = M\Phi_3$$

(2.6)

$$\begin{aligned}
-i\epsilon\Phi_1 + \gamma\frac{\partial \Phi_0}{dr} + \gamma\frac{\nu}{\sinh r}\Phi_0 &= ME_1\ , \\
-i\epsilon\Phi_2 &= ME_2\ , \\
-i\epsilon\Phi_3 - \gamma\frac{\partial \Phi_0}{\partial r} + \gamma\frac{\nu}{\sinh r}\Phi_0 &= ME_3\ ,
\end{aligned} \tag{2.7}$$

$$\begin{gathered}
-i\gamma\frac{\partial \Phi_2}{\partial r} - i\gamma\frac{\nu}{\sinh r}\Phi_2 = M\cosh z H_1\ , \\
-i\gamma(\frac{\partial \Phi_1}{\partial r} + \frac{\partial \Phi_3}{\partial r}) + \frac{i\gamma}{\sinh r}[(\nu - \cosh r)\Phi_1 - (\nu + \cosh r)\Phi_3] = MH_2\ , \\
-i\gamma\frac{\partial \Phi_2}{\partial r} + i\gamma\frac{\nu}{\sinh r}\Phi_2 = MH_3\ .
\end{gathered} \tag{2.8}$$

With the notation

$$\frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu - \cosh r}{\sinh r}) = \hat{a}_-,\ \frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu + \cosh r}{\sinh r}) = \hat{a}_+\ ,\ \frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu}{\sinh r}) = \hat{a}\ ,$$

$$\frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu - \cosh r}{\sinh r}) = \hat{b}_-,\ \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu + \cosh r}{\sinh r}) = \hat{b}_+\ ,\ \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu}{\sinh r}) = \hat{b}\ ,$$

the above system reads

$$\begin{aligned}
-\hat{b}_- E_1 - \hat{a}_+ E_3 &= M\ \Phi_0\ , \\
-i\hat{b}_- H_1 + i\hat{a}_+\ H_3 + i\epsilon\ E_2 &= M\ \Phi_2\ , \\
i\hat{a} H_2 + i\epsilon\ E_1 &= M\ \Phi_1\ , \\
-i\hat{b} H_2 + i\epsilon\ E_3 &= M\ \Phi_3\ ,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\hat{a}\Phi_0 - i\epsilon\ \Phi_1 &= M\ E_1\ , \\
-i\hat{a}\Phi_2 &= M\ H_1\ , \\
\hat{b}\Phi_0 - i\epsilon\ \Phi_3 &= M\ E_3\ , \\
i\hat{b}\Phi_2 &= M\ H_3\ , \\
-i\epsilon\Phi_2 &= M\ E_2\ , \\
i\hat{b}_-\Phi_1 - i\hat{a}_+\Phi_3 &= M\ H_2\ .
\end{aligned} \tag{2.10}$$

3. Nonrelativistic approximation

Excluding non-dynamical variables Φ_0, H_1, H_2, H_3 with the help of equations

$$
\begin{aligned}
-\hat{b}_- E_1 - \hat{a}_+ E_3 = M\ \Phi_0\ , \\
-i\ \hat{a}\Phi_2 = M\ H_1\ , \\
i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3 = M\ H_2\ , \\
i\ \hat{b}\Phi_2 = M\ H_3\ ,
\end{aligned}
\tag{3.1}
$$

we get 6 equations (grouping them in pairs)

$$
\begin{aligned}
i\hat{a}\ (i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3) + i\epsilon\ ME_1 = M^2 \Phi_1\ , \\
\hat{a}\ (-\hat{b}_- E_1 - \hat{a}_+ E_3 - i\epsilon\ M\Phi_1 = M^2 e_1\ ,
\end{aligned}
\tag{3.2a}
$$

$$
\begin{aligned}
-i\hat{b}_-\ (-i\ \hat{a}\ \Phi_2) + i\hat{a}_+ (i\ \hat{b}\ \Phi_2) + i\epsilon\ ME_2 = M^2 \Phi_2\ , \\
-i\epsilon\ M\Phi_2 = M^2 E_2\ ,
\end{aligned}
\tag{3.2b}
$$

$$
\begin{aligned}
-i\ \hat{b}\ (i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3) + i\epsilon\ ME_3 = M^2 \Phi_3\ , \\
\hat{b}\ (-\hat{b}_- E_1 - \hat{a}_+ E_3) - i\epsilon\ M\Phi_3 = M^2 E_3\ ,
\end{aligned}
\tag{3.2c}
$$

Now we introduce big and small constituents

$$
\begin{aligned}
\Phi_1 = \Psi_1 + \psi_1\ , \qquad iE_1 = \Psi_1 - \psi_1\ , \\
\Phi_2 = \Psi_2 + \psi_2\ , \qquad iE_2 = \Psi_2 - \psi_2\ , \\
\Phi_3 = \Psi_3 + \psi_3\ , \qquad iE_3 = \Psi_3 - \psi_3\ ;
\end{aligned}
$$

besides we should separate the rest energy by formal change $\epsilon \Longrightarrow \epsilon + M$; summing and subtracting equation within each pair (3.2) and ignoring small constituents ψ_i we arrive at three equations for big components

$$
\begin{aligned}
\Big(-2\ \hat{a}\hat{b}_- + 2\epsilon M \Big)\ \Psi_1 = 0\ , \\
\Big(-(\hat{b}_- \hat{a} + \hat{a}_+ \hat{b}) + 2\epsilon M \Big)\ \Psi_2 = 0\ , \\
\Big(-2\hat{b}\hat{a}_+ + 2\epsilon M \Big)\ \Psi_3 = 0\ .
\end{aligned}
\tag{3.4}
$$

It is a needed Pauli-like system for the spin 1 particle.

Explicitly they read

$$
\left[\frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r}\ \frac{d}{dr} - \frac{1}{\sinh r}\ \frac{d\nu}{dr} - \frac{1 - 2\,\nu\ \cosh r}{\sinh^2 r} - \frac{\nu^2}{\sinh^2 r} + 2\,\epsilon\, M \right] \Psi_1 = 0\,,
$$

$$\left[\frac{d^2}{dr^2}+\frac{\cosh r}{\sinh r}\frac{d}{dr}-\frac{\nu^2}{\sinh^2 r}+2\,\epsilon\, M\right]\Psi_2=0\,,$$

$$\left[\frac{d^2}{dr^2}+\frac{\cosh r}{\sinh r}\frac{d}{dr}+\frac{1}{\sinh r}\frac{d\nu}{dr}-\frac{1+2\,\nu\,\cosh r}{\sinh^2 r}-\frac{\nu^2}{\sinh^2 r}+2\,\epsilon\, M\right]\Psi_3=0\,. \tag{3.5}$$

Allowing for $\nu(r)=m+B\,(\cosh r-1)$ we arrive at

$$\left[\frac{d^2}{dr^2}+\frac{\cosh r}{\sinh r}\frac{d}{dr}-B-\frac{1-2\,[m+B\,(\cosh r-1)]\,\cosh r}{\sinh^2 r}-\right.$$

$$\left.-\frac{[m+B\,(\cosh r-1)]^2}{\sinh^2 r}+2\,\epsilon\, M\right]\Psi_1=0\,,$$

$$\left[\frac{d^2}{dr^2}+\frac{\cosh r}{\sinh r}\frac{d}{dr}-\frac{[m+B\,(\cosh r-1)]^2}{\sinh^2 r}+2\,\epsilon\, M\right]\Psi_2=0\,,$$

$$\left[\frac{d^2}{dr^2}+\frac{\cosh r}{\sinh r}\frac{d}{dr}+B-\frac{1+2\,[m+B\,(\cosh r-1)]\,\cosh r}{\sinh^2 r}-\right.$$

$$\left.-\frac{[m+B\,(\cosh r-1)]^2}{\sinh^2 r}+2\,\epsilon\, M\right]\Psi_3=0\,. \tag{3.6}$$

The first and the third equations are symmetric with respect to formal change $m\Longrightarrow -m\,,\ B\Longrightarrow -B$.

In the new variable $1-\cosh r=2\,y$, they look

$$y\,(1-y)\,\frac{d^2\Psi_1}{dy^2}+(1-2\,y)\,\frac{dB_1}{dy}+$$

$$+\left[B^2-B-2\,\epsilon\, M-\frac{1}{4}\,\frac{(2\,B-m-1)^2}{1-y}-\frac{1}{4}\,\frac{(m-1)^2}{y}\right]\Psi_1=0\,, \tag{3.7a}$$

$$y\,(1-y)\,\frac{d^2\Psi_2}{dy^2}+(1-2\,y)\,\frac{dB_2}{dy}+$$

$$+\left[B^2-2\,\epsilon\, M-\frac{1}{4}\,\frac{(2\,B-m)^2}{1-y}-\frac{1}{4}\,\frac{m^2}{y}\right]\Psi_2=0\,, \tag{3.7b}$$

$$y\,(1-y)\,\frac{d^2\Psi_3}{dy^2}+(1-2\,y)\,\frac{dB_3}{dy}+$$

$$+\left[B^2 + B - 2\,\epsilon\, M - \frac{1}{4}\,\frac{(2\,B - m + 1)^2}{1-y} - \frac{1}{4}\,\frac{(m+1)^2}{y}\right]\Psi_3 = 0\,. \tag{3.7c}$$

Eq. (3.7a) with the substitution

$$\Psi_1 = y^{C_1}(1-y)^{A_1} f_1$$

leads to

$$y\,(1-y)\,\frac{d^2\Psi_1}{dy^2} + [2\,C_1 + 1 - (2\,A_1 + 2\,C_1 + 2)\,y]\,\frac{dB_1}{dy} +$$
$$+\big[B^2 - B - 2\,\epsilon\, M - (A_1 + C_1)\,(A_1 + C_1 + 1) +$$
$$+\frac{1}{4}\,\frac{4\,A_1^2 - (2\,B - m - 1)^2}{1-y} + \frac{1}{4}\,\frac{4\,C_1^2 - (m-1)^2}{y}\Big]\,\Psi_1 = 0\,. \tag{3.8}$$

At A_1, C_1 obeying

$$A_1 = \pm\frac{1}{2}\,(2\,B - m - 1)\,, \qquad C_1 = \pm\frac{1}{2}\,(m-1)\,,$$

eq. (3.8) becomes simpler

$$y\,(1-y)\,\frac{d^2\Psi_1}{dy^2} + [2\,C_1 + 1 - (2\,A_1 + 2\,C_1 + 2)\,y]\,\frac{dB_1}{dy} +$$
$$+\big[B^2 - B - 2\,\epsilon\, M - (A_1 + C_1)\,(A_1 + C_1 + 1)\big]\,\Psi_1 = 0\,, \tag{3.9a}$$

what is hypergeometric equation with parameters

$$\alpha_1 = A_1 + C_1 + \frac{1}{2} + \sqrt{B^2 - B - 2\,\epsilon\, M + \frac{1}{4}}\,,$$
$$\beta_1 = A_1 + C_1 + \frac{1}{2} - \sqrt{B^2 - B - 2\,\epsilon\, M + \frac{1}{4}}\,,$$
$$\gamma_1 = 2\,C_1 + 1\,. \tag{3.9b}$$

To have finite and single-valued solutions one must impose restrictions $A_1 < 0$, $C_1 > 0$. Besides, one must get n-order polynomials and satisfy the inequality $A_1 + C_1 + n < 0$.

Four different possibilities for A_1, C_1 are (for definiteness let it be $B > 0$):

1. $A_1 = -\frac{1}{2}\,(2\,B - m - 1)\,, \qquad C_1 = -\frac{1}{2}\,(m-1)\,,$

2. $A_1 = +\frac{1}{2}\,(2\,B - m - 1)\,, \qquad C_1 = -\frac{1}{2}\,(m-1)\,,$

3. $A_1 = +\frac{1}{2}\,(2\,B - m - 1)\,, \qquad C_1 = +\frac{1}{2}\,(m-1)\,,$

$$4. \qquad A_1 = -\frac{1}{2}(2B - m - 1), \qquad C_1 = +\frac{1}{2}(m-1).$$

To describe bound state, only variants 1 and 4 are appropriate:

$$1 \ , \qquad m < 0 ,$$

$$\alpha_1 = -B + \frac{3}{2} + \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} ,$$

$$\beta_1 = -B + \frac{3}{2} - \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} ,$$

$$\gamma_1 = -m + 2 ,$$

$$\text{spectrum} \qquad \alpha_1 = -n , \qquad \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} = B - \frac{3}{2} - n , \qquad (3.10a)$$

$$\epsilon M = B - 1 + n \left(B - \frac{3}{2} - \frac{n}{2} \right) ;$$

$$4 \ , \qquad 0 < m < B ,$$

$$\alpha_1 = -B + m + \frac{1}{2} + \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} ,$$

$$\beta_1 = -B + m + \frac{1}{2} - \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} ,$$

$$\gamma_1 = m ,$$

$$\text{spectrum} \qquad \alpha_1 = -n , \qquad \sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} = B - \frac{1}{2} - (n+m) , \qquad (3.10b)$$

$$\epsilon M = (m+n) \left(B - \frac{1}{2} - \frac{1}{2}(m+n) \right) .$$

Formulas (3.10a,b) can be jointed into single one

$$\sqrt{B^2 - B - 2\epsilon M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{|\, 2B - m - 1\,| + |\, m - 1\,|}{2} . \qquad (3.10c)$$

From eq. $(3.7b)$ with the substitution

$$\Psi_2 = y^{C_2}(1-y)^{A_2} f_2$$

we get

$$y(1-y)\frac{d^2 f_2}{dy^2} + [2C_2 + 1 - (2A_2 + 2C_2 + 2)\,y]\frac{df_2}{dy} +$$

$$+\left[B^2 - 2\,\epsilon\, M - (A_2 + C_2)\,(A_2 + C_2 + 1) + \right.$$

$$\left. + \frac{1}{4}\,\frac{4\,A_2^2 - (2\,B - m)^2}{1-y} + \frac{1}{4}\,\frac{4\,C_2^2 - m^2}{y}\right] f_2 = 0\,. \tag{3.11}$$

At

$$A_2 = \pm\frac{1}{2}\,(2\,B - m)\,, \qquad C_2 = \pm\frac{m}{2}\,,$$

eq. (3.11) becomes simpler

$$y\,(1-y)\,\frac{d^2 f_2}{dy^2} + [2\,C_2 + 1 - (2\,A_2 + 2\,C_2 + 2)\,y]\,\frac{df_2}{dy} +$$

$$+\left[B^2 - 2\,\epsilon\, M - (A_2 + C_2)\,(A_2 + C_2 + 1)\right] f_2 = 0 \tag{3.12a}$$

which is recognized as of hypergeometric type

$$\alpha_2 = A_2 + C_2 + \frac{1}{2} + \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}}\,,$$

$$\beta_2 = A_2 + C_2 + \frac{1}{2} - \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}}\,,$$

$$\gamma_2 = 2\,C_2 + 1\,. \tag{3.12b}$$

From four variants

$$\begin{array}{lllll}
1. & A_2 = -\frac{1}{2}\,(2\,B - m)\,, & & C_2 = -\frac{m}{2}\,, \\
2. & A_2 = +\frac{1}{2}\,(2\,B - m)\,, & & C_2 = -\frac{m}{2}\,, \\
3. & A_2 = +\frac{1}{2}\,(2\,B - m)\,, & & C_2 = +\frac{m}{2}\,, \\
4. & A_2 = -\frac{1}{2}\,(2\,B - m)\,, & & C_2 = +\frac{m}{2}
\end{array}$$

only 1 and 4 seem to be approptiate to describe bound states:

$$1\ , \qquad m < 0\,,$$

$$\alpha_2 = -B + \frac{1}{2} + \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}}\,,$$

$$\beta_2 = -B + \frac{1}{2} - \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}}\,,$$

$$\gamma_2 = -m + 1\,,$$

$$\text{spectrum} \qquad \alpha_2 = -n\,, \qquad \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} = B - \frac{1}{2} - n\,, \tag{3.13a}$$

$$\epsilon M = \frac{B}{2} + n \left(B - \frac{1}{2} - \frac{n}{2} \right) ;$$

$$4 \;\; , \qquad 0 < m < B \, ,$$

$$\alpha_2 = -B + m + \frac{1}{2} + \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} \, ,$$

$$\beta_2 = -B + m + \frac{1}{2} - \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} \, ,$$

$$\gamma_2 = m + 1 \, ,$$

$$\text{spectrum} \qquad \alpha_2 = -n \, , \qquad \sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} = B - \frac{1}{2} - (n+m) \, , \tag{3.13b}$$

$$\epsilon M = \frac{B}{2} + (m+n) \left(B - \frac{1}{2} - \frac{1}{2}\,(m+n) \right) .$$

Formulas (3.13a,b) can be joint into a single one

$$\sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{\mid 2B - m \mid + \mid m \mid}{2} \, . \tag{3.13c}$$

The region for allowed values of m for bound states can be illustrated by Fig. 1.

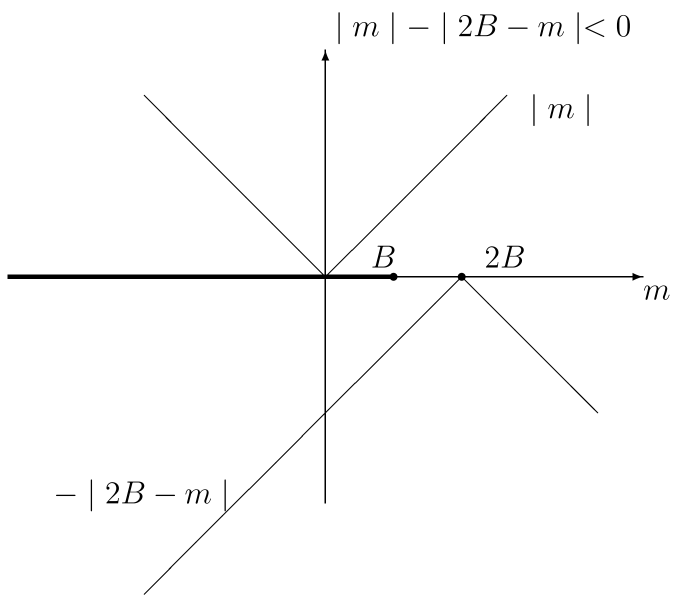

Fig. 1. Bound states at $B > 0: \; m < B$

At $B < 0$, we shoul have different Fig. . 2.

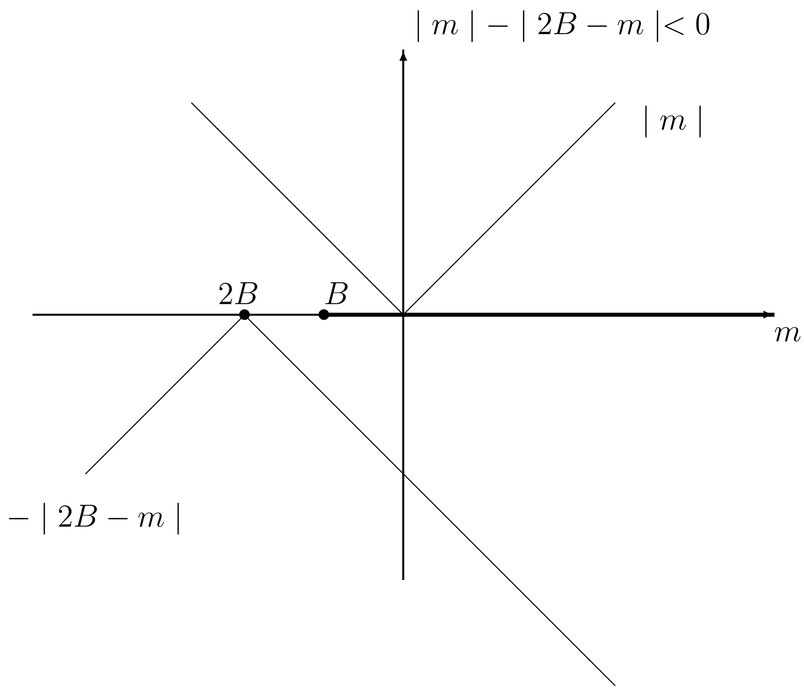

Fig. 2. Bound states at $B < 0 : \; B < m$

Similar Figures can be given in connection with the functions $\Psi_1(y)$ and Ψ_3 as well. In case of (3.7c), with substitution

$$\Psi_3 = y^{C_3} (1-y)^{A_3} f_3 \, ,$$

we will obtain

$$y\,(1-y)\,\frac{d^2 f_3}{dy^2} + [2\,C_3 + 1 - (2\,A_3 + 2\,C_3 + 2)\,y]\,\frac{df_3}{dy} +$$
$$+ \Big[B^2 + B - 2\,\epsilon\,M - (A_3 + C_3)\,(A_3 + C_3 + 1) +$$
$$+ \frac{1}{4}\,\frac{4\,A_3^2 - (2\,B - m + 1)^2}{1-y} + \frac{1}{4}\,\frac{4\,C_3^2 - (m+1)^2}{y}\Big]\,f_3 = 0 \, . \qquad (3.14)$$

At A_3, C_3

$$A_3 = \pm\,\frac{1}{2}\,(2\,B - m + 1)\,, \qquad C_3 = \pm\,\frac{1}{2}\,(m+1)\,,$$

eq. (3.14) will read

$$y\,(1-y)\,\frac{d^2 f_3}{dy^2} + [2\,C_3 + 1 - (2\,A_3 + 2\,C_3 + 2)\,y]\,\frac{df_3}{dy} +$$

$$+\left[B^2+B-2\,\epsilon\,M-(A_3+C_3)\,(A_3+C_3+1)\right]f_3=0 \tag{3.15a}$$

that is a hypergeometric equation

$$\alpha_3=A_3+C_3+\frac{1}{2}+\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\beta_3=A_3+C_3+\frac{1}{2}-\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\gamma_3=2\,C_3+1\,. \tag{3.15b}$$

From four possibilities

$$
\begin{array}{llll}
1. & A_3 \;=\; -\dfrac{1}{2}\,(2\,B-m+1)\,, & & C_3 \;=\; -\dfrac{1}{2}\,(m+1)\,,\\
2. & A_3 \;=\; +\dfrac{1}{2}\,(2\,B-m+1)\,, & & C_3 \;=\; -\dfrac{1}{2}\,(m+1)\,,\\
3. & A_3 \;=\; +\dfrac{1}{2}\,(2\,B-m+1)\,, & & C_3 \;=\; +\dfrac{1}{2}\,(m+1)\,,\\
4. & A_3 \;=\; -\dfrac{1}{2}\,(2\,B-m+1)\,, & & C_3 \;=\; +\dfrac{1}{2}\,(m+1)\,.
\end{array}
$$

only 1 and 4 are appropriate to describe bound states:

$$1\quad,\qquad m\;<0\,,$$

$$\alpha_3=-B-\frac{1}{2}+\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\beta_3=-B-\frac{1}{2}-\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\gamma_3=-m\,,$$

$$\text{spectrum}\qquad \alpha_3=-n\,,\qquad \sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}=B+\frac{1}{2}-n\,, \tag{3.16a}$$

$$\epsilon\,M=n\,\left(B+\frac{1}{2}-\frac{n}{2}\right)\;;$$

$$4\quad,\qquad 0<m<B\,,$$

$$\alpha_3=-B+m+\frac{1}{2}+\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\beta_3=-B+m+\frac{1}{2}-\sqrt{B^2+B-2\,\epsilon\,M+\frac{1}{4}}\,,$$

$$\gamma_3=m+2\,,$$

$$\text{spectrum} \qquad \alpha_3 = -n\,, \qquad \sqrt{B^2 + B - 2\,\epsilon\, M + \frac{1}{4}} = B - \frac{1}{2} - (n+m)\,, \qquad (3.16b)$$

$$\epsilon\, M = B + (m+n)\,\left(B - \frac{1}{2} - \frac{1}{2}\,(m+n)\right)\,.$$

Again, formulas (3.16a,b) can be joint into a single one

$$\sqrt{B^2 + B - 2\,\epsilon\, M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{|\,2B - m + 1\,| + |\,m + 1\,|}{2}\,. \qquad (3.16c)$$

4. Solution of radial equations in relativistic case

Let start with eqs. (2.4)–(2.5)

$$
\begin{gathered}
-\hat{b}_- E_1 - \hat{a}_+ E_3 = M\ \Phi_0\ , \\
-i\hat{b}_- H_1 + i\hat{a}_+\ H_3 + i\epsilon\ E_2 = M\ \Phi_2\ , \\
i\hat{a} H_2 + i\epsilon\ E_1 = M\ \Phi_1\ , \\
-i\hat{b} H_2 + i\epsilon\ E_3 = M\ \Phi_3\ , \qquad (4.1)\\
\hat{a}\Phi_0 - i\epsilon\ \Phi_1 = M\ E_1\ , \\
-i\hat{a}\Phi_2 = M\ H_1\ , \\
\hat{b}\Phi_0 - i\epsilon\ \Phi_3 = M\ E_3\ , \\
i\hat{b}\Phi_2 = M\ H_3\ . \\
-i\epsilon\Phi_2 = M\ E_2\ , \\
i\hat{b}_-\Phi_1 - i\hat{a}_+\Phi_3 = M\ H_2\ , \qquad (4.2)
\end{gathered}
$$

Excluding six components E_i, H_i, we derive four second order equations for Φ_a:

$$
\begin{gathered}
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0\ , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon(\hat{b}_-\Phi_1 + \hat{a}_+\Phi_3) = 0\ , \\
(-\hat{a}\hat{b}_- + \epsilon^2 - M^2)\Phi_1 + \hat{a}\hat{a}_+\Phi_3 + i\epsilon\hat{a}\Phi_0 = 0\ , \\
(-\hat{b}\hat{a}_+ + \epsilon^2 - M^2)\Phi_3 + \hat{b}\hat{b}_-\Phi_1 + i\epsilon\hat{b}\Phi_0 = 0\ . \qquad (4.3)
\end{gathered}
$$

Once, it should be noted existence of a simple solution of the system

$$
\begin{gathered}
\Phi_0 = 0\ , \qquad \Phi_1 = 0\ , \qquad \Phi_3 = 0\ , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0\ . \qquad (4.4a)
\end{gathered}
$$

and simple expressions for tensors components

$$E_1 = 0\ , \qquad H_1 = -iM^{-1}\hat{a}\ \Phi_2\ ,$$

$$E_3 = 0 \; , \qquad H_3 = iM^{-1}\hat{b}\; \Phi_2 \; ,$$
$$E_2 = -i\epsilon M^{-1}\Phi_2 \; , \qquad H_2 = 0 \; . \tag{4.4b}$$

Lets us turn to (4.3) and act on the third equation from the left by operator $\hat{b}_-$, and on the forth equation by operator $\hat{a}_+$. Thus, introducing the notation

$$\hat{b}_-\Phi_1 = Z_1 \; , \qquad \hat{a}_+\Phi_3 = Z_3 \; ,$$

instead of (4.3) we obtain

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; ,$$
$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon(Z_1 + Z_3) = 0 \; ,$$
$$(-\hat{b}_-\hat{a} + \epsilon^2 - M^2)Z_1 + \hat{b}_-\hat{a}Z_3 + i\epsilon\hat{b}_-\hat{a}\Phi_0 = 0 \; ,$$
$$(-\hat{a}_+\hat{b} + \epsilon^2 - M^2)Z_3 + \hat{a}_+\hat{b}Z_1 + i\epsilon\hat{a}_+\hat{b}\Phi_0 = 0 \; . \tag{4.5}$$

Instead of Z_1, Z_3, let us introduce new functions

$$Z_1 = \frac{f+g}{2} \; , \qquad Z_3 = \frac{f-g}{2} \; ,$$
$$Z_1 + Z_3 = f \; , \qquad Z_1 - Z_3 = g \; ;$$

the the above system reads

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; ,$$
$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon\; f = 0 \; ,$$
$$-\hat{b}_-\hat{a}\frac{f+g}{2} + (\epsilon^2 - M^2)\frac{f+g}{2} + \hat{b}_-\hat{a}\frac{f-g}{2} + i\epsilon\hat{b}_-\hat{a}\Phi_0 = 0 \; ,$$
$$-\hat{a}_+\hat{b}\frac{f-g}{2} + (\epsilon^2 - M^2)\frac{f-g}{2} + \hat{a}_+\hat{b}\frac{f+g}{2} + i\epsilon\hat{a}_+\hat{b}\Phi_0 = 0 \; . \tag{4.6}$$

After elementary manipulations with equation 3 and 4 we get

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; ,$$
$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon\; f = 0 \; ,$$
$$-\hat{b}_-\hat{a}g + (\epsilon^2 - M^2)\frac{f+g}{2} + i\epsilon\hat{b}_-\hat{a}\Phi_0 = 0 \; ,$$
$$\hat{a}_+\hat{b}g + (\epsilon^2 - M^2)\frac{f-g}{2} + i\epsilon\hat{a}_+\hat{b}\Phi_0 = 0 \; .$$

Now, summing and subtracting equations 3 and 4, we obtain

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; ,$$
$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon\; f = 0 \; ,$$

$$(-\hat{b}_-\hat{a}+\hat{a}_+\hat{b})g+(\epsilon^2-M^2)f+i\epsilon(\hat{b}_-\hat{a}+\hat{a}_+\hat{b})\ \Phi_0=0\ ,$$
$$(-\hat{b}_-\hat{a}-\hat{a}_+\hat{b})g+(\epsilon^2-M^2)g+i\epsilon(\hat{b}_-\hat{a}-\hat{a}_+\hat{b})\Phi_0=0\ , \qquad (4.7)$$

Taking into account identities

$$-\hat{b}_-\hat{a}-\hat{a}_+\hat{b}=\Delta_2=...$$
$$-\hat{b}_-\hat{a}+\hat{a}_+\hat{b}=2B \qquad (4.8)$$

we arrive at the system

$$(\Delta_2+\epsilon^2-M^2)\ \Phi_2=0\ , \qquad (4.9)$$

$$(\Delta_2-M^2)\ \Phi_0+i\epsilon\ f=0\ ,$$
$$2B\ g+(\epsilon^2-M^2)f-i\epsilon\Delta_2\ \Phi_0=0\ ,$$
$$\Delta_2 g+(\epsilon^2-M^2)g-2i\epsilon B\ \Phi_0=0\ , \qquad (4.10)$$

From the second equation, with the use of expression for $\Delta_2\Phi_0$ according to the first equation, we derive linear relation between three functions

$$2B\ g-M^2 f-i\epsilon M^2\Phi_0=0\ . \qquad (4.11)$$

Let us exclude f

$$f=\frac{2B}{M^2}\ g-i\epsilon\Phi_0$$

so we get

$$(\Delta_2+\epsilon^2-M^2)\ g=2i\epsilon B\ \Phi_0\ ,$$
$$(\Delta_2+\epsilon^2-M^2)\ \Phi_0=-\frac{2i\epsilon B}{M^2}\ g. \qquad (4.12)$$

With notation $\gamma=\epsilon^2/M^2$, the system can be presented in a matrix form as follows

$$(\Delta_2+\epsilon^2-M^2)\left|\begin{array}{r} g \\ \epsilon\ \Phi_0 \end{array}\right| = \left|\begin{array}{rr} 0 & 2iB \\ -2iB\gamma & 0 \end{array}\right| \left|\begin{array}{r} g \\ \epsilon\ \Phi_0 \end{array}\right|. \qquad (4.13)$$

or symbolically

$$\Delta f=Af \qquad \Delta f'=SAS^{-1}\ f'\ , \qquad f'=Sf\ .$$

It remains to find a transformation reducing the matrix A to a diagonal form

$$SAS^{-1}=\left|\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right|, \qquad S=\left|\begin{array}{cc} a & d \\ c & b \end{array}\right|\ ;$$

the problem is equivalent to the linear system

$$-\lambda_1\ a-2i\gamma B\ d=0\ ,$$

$$2iB\ a-\lambda_1\ d=0\ ;$$

$$-\lambda_2 \, c - 2i\gamma B \, b = 0 \, ,$$
$$2iB \, c - \lambda_2 \, b = 0 \, .$$

Its solutions can be chosen in the form

$$\lambda_1 = +\frac{2\epsilon B}{M} \, , \qquad \lambda_2 = -\frac{2\epsilon B}{M} \, ,$$

$$S = \left| \begin{array}{cc} \epsilon & +iM \\ \epsilon & -iM \end{array} \right| , \qquad S^{-1} = \frac{1}{-2i\epsilon M} \left| \begin{array}{cc} -iM & -iM \\ -\epsilon & \epsilon \end{array} \right| . \tag{4.14}$$

New (primed) function satisfy the following equations

$$1) \qquad \left(\Delta_2 + \epsilon^2 - M^2 - \frac{2\epsilon B}{M} \right) \, g' = 0 \, , \tag{4.15a}$$

$$2) \qquad \left(\Delta_2 + \epsilon^2 - M^2 + \frac{2\epsilon B}{M} \right) \, \Phi_0' = 0 \, . \tag{4.15b}$$

they are independent from each other, therefore there exist two solutions

$$1) \qquad g' \neq 0, \qquad \Phi_0' = 0 \, , \tag{4.16a}$$

$$2) \qquad g' = 0, \qquad \Phi_0' \neq 0 \, . \tag{4.16b}$$

The initial functions for these two cases assume respectively the form

$$g = \frac{1}{2\epsilon} g' + \frac{1}{2i\epsilon} \epsilon \Phi_0' \, , \qquad \epsilon \Phi_0 = \frac{1}{2iM} g' - \frac{1}{2iM} \epsilon \Phi_0' \, . \tag{4.17}$$

In cases 1) and 2) they assume respectively the form

$$1) \qquad g = \frac{1}{2\epsilon} g' \, , \qquad \epsilon \Phi_0 = \frac{1}{2iM} g' \, . \tag{4.18a}$$

$$2) \qquad g = \frac{1}{2i\epsilon} \epsilon \Phi_0' \, , \qquad \epsilon \Phi_0 = -\frac{1}{2iM} \epsilon \Phi_0' \, . \tag{4.18b}$$

To obtain explicit solutions for these differential equation, we need not any additional calculations, instead it suffices to perform simple formal changes as pointed below

$$\left[\frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - \frac{[m + B\,(\cosh r - 1)]^2}{\sinh^2 r} + 2\,\epsilon\, M \right] f(r) = 0 \, ,$$
$$\sqrt{B^2 - 2\,\epsilon\, M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{\mid 2B - m \mid + \mid m \mid}{2} \tag{4.19}$$

$$2\,\epsilon\, M \qquad \Longrightarrow \qquad \left\{ \begin{array}{lll} (\epsilon^2 - M^2 - \frac{2\epsilon B}{M}) & --- & (4.9) \\ (\epsilon^2 - M^2) & --- & (4.15a) \\ (\epsilon^2 - M^2 + \frac{2\epsilon B}{M}) & --- & (4.15b) \end{array} \right. \tag{4.20}$$

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